International Journal of Theoretical Physics, Vol. 32, No. 2, 1993

Physical Continuum and the Problem of a Finitistic Quantum Field Theory

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Received February 11, 1992

To formulate a finitistic quantum field theory, the hypothesis is made that the continuum of space and time is countable possessing the cardinal number $\aleph_0$. With the integers having the same cardinal number, it is therefore assumed that distances in space and time can be expressed only in integer multiples of a fundamental length and time. To preserve the condition of causality, a quantized field theory derived under this assumption must be expressed in absolute space and time, with the field equation invariant under Galilei transformations. It is shown that such a theory not only can be formulated in full agreement with all the postulates of quantum mechanics, but that it leads to Lorentz invariance as a dynamic symmetry in the limit of low energies. If the smallest length and time are chosen to be equal to the Planck length and time, respectively, observable departures from the predictions of special relativity would become effective only in approaching the Planck energy of $\sim 10^{19}$ GeV.

1. INTRODUCTION

According to transfinite set theory, all countable numbers have the cardinal number $\aleph_0$. Because the counting is done by the natural numbers, they, too, have the cardinal number $\aleph_0$. The countable numbers include all rational numbers. But they also include all those irrational numbers which are roots of algebraic equations with rational coefficients. Because all these numbers can be obtained by a countable set of rules, they are accessible to a computer. These numbers densely cover the continuum of space and time, and for this reason can describe all of physical reality. However, in mathematics there are infinitely more noncountable numbers than countable numbers; these are the transcendental numbers like π and e. They cover the continuum even more densely and they have the much larger cardinal

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0020-7748/93/0200-0261$07.00/0

number $\aleph_1$, which according to Cantor's continuum hypothesis is given by

$$\aleph_1 = 2^{\aleph_0} \tag{1.1}$$

A differential equation used as a model to describe physical reality covers all mathematical numbers, not only those belonging to $\aleph_0$, but also those belonging to $\aleph_1$. Therefore, if physical reality is restricted to those numbers having the cardinal number $\aleph_0$, and which are also accessible to a computer, a differential equation would in principle be unsuitable to describe physical reality. A lattice space with difference equations on the lattice replacing the differential equations in the continuum is also unsuitable because such a space eliminates the continuum of densely spaced points and has a discrete rather than a continuous group of translations and rotations. In accordance with the assumption that physical space and time have the cardinal number $\aleph_0$, we may postulate instead that all distances in space and time can be measured only in integer multiples of a fundamental length and fundamental time, with the integer multiples given by the natural numbers and which have the cardinal number $\aleph_0$.

2. FINITE-DIFFERENCE OPERATORS

Replacing differentials by finite differences, we follow a procedure outlined by Madelung (1950), expressing the finite-difference quotient of a function $y=f(x)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+l/2)-f(x-l/2)}{l} \tag{2.1}$$

through the finite-difference operator

$$f(x+h) = e^{hd/dx} f(x) = f(x) + h\frac{df(x)}{dx} + \frac{h^2}{2!}\frac{d^2f(x)}{dx^2} + \cdots \tag{2.2}$$

With $h=l/2$, we find

$$\frac{\Delta y}{\Delta x} = \frac{\sinh[(l/2)\, d/dx]}{l/2} f(x) \tag{2.3}$$

In a similar way we can define a Riemann integral average y value $\bar{y}$ by

$$\bar{y} = [f(x+l/2)+f(x-l/2)]/2 = \cosh[(l/2)\, d/dx]\, f(x) \tag{2.4}$$

In the limit $l \to 0$, (2.3) becomes dy/dx and (2.4) becomes y. Putting $d/dx = \partial$,we may introduce the operators

$$\begin{aligned} \Delta_0 &= \cosh[(l/2)\partial] \\ \Delta_1 &= (2/l)\sinh[(l/2)\partial] \end{aligned} \tag{2.5}$$

such that

$$\frac{\Delta y}{\Delta x} = \Delta_1 f(x)$$
$$\bar{y} = \Delta_0 f(x) \tag{2.6}$$

and furthermore

$$\Delta_1 = (2/l)^2\, d\Delta_0/d\partial \tag{2.7}$$

Both operators are solutions of

$$[d^2/d\partial^2 - (l/2)^2]\Delta(\partial) = 0 \tag{2.8}$$

To obtain the difference operator in an N-dimensional space, we generalize the differential equation for the one-dimensional "average value" operator $\Delta_0(\partial)$

$$[d^2/d\partial^2 - (l/2)^2]\Delta_0(\partial) = 0 \tag{2.9}$$

to an N-dimensional space, where $\Delta_0(\partial)$ must obey the condition

$$\lim_{l\to 0} \Delta_0(\partial) = 1 \tag{2.10}$$

Calling the generalized N-dimensional "average value" operator Δ_0^N, it would have to satisfy the partial differential equation

$$\left[\sum_{i=1}^{N} \frac{\partial^2}{\partial(\partial_i)^2} - N\left(\frac{l}{2}\right)^2\right]\Delta_0^N = 0 \tag{2.11}$$

where $\partial_i = \partial/\partial x_i$, and where $\lim_{l\to 0} \Delta_0^N = 1$.

Because Δ_0^N must be a scalar, and because the only vector invariant which is a scalar is

$$\partial = \left(\sum_{i=1}^{N} \partial_i^2\right)^{1/2} \tag{2.12}$$

one must have $\Delta_0^N = \Delta_0^N(\partial)$. One can therefore introduce into (2.11) N-dimensional polar coordinates, and one obtains the ordinary differential equation

$$\left[\frac{1}{\partial^{N-1}}\frac{d}{d\partial}\left(\partial^{N-1}\frac{d}{d\partial}\right) - N\left(\frac{l}{2}\right)^2\right]\Delta_0^N(\partial) = 0 \tag{2.13}$$

The general solution of (2.13) can be expressed in terms of cylinder functions (Kamke, 1959). Having obtained the scalar operator function

$\Delta_0^N(\partial)$, one finds the generalized difference operator invariant under arbitrary translations and rotations as

$$\Delta_1^N = (2/l)^2 \, d\Delta_0^N / d\partial_i \tag{2.14}$$

leading for $N=1$ to (2.7).

With the operators Δ_0^N and Δ_1^N one is in a position to translate any differential equation of mathematical physics into a finitistic form. The operators Δ_0^N and Δ_1^N involve differentials of infinite order, which means that this translation leads to differential equations of infinite order. Because these infinite-order differential equations somehow belong to $\aleph_1$, we can see the close mathematical relationship between the countable and noncountable.

3. CAUSALITY AND LORENTZ INVARIANCE

Introducing a fundamental length into a relativistic quantum field theory can eliminate the divergences, but it also leads to a violation of causality. Because the metric of the Minkowski space-time is not positive definite, the property of proximity between two points in space cannot be formulated in a relativistically invariant way. As a solution to this problem, we offer the idea that the fundamental kinematic symmetry of nature is the Galilei group, with the Lorentz group a derived dynamic symmetry, with the latter valid only in the asymptotic limit of low energies. This idea suggests the proposal of the existence of a fundamental field described by an exactly nonrelativistic Heisenberg-type nonlinear field equation. As in Heisenberg's theory, the elementary particles and their interactions would have to be derived from this field. To avoid the divergences of relativistic quantum field theories, a cutoff length can be introduced without violating causality. If the fundamental field has collective excitations obeying the classical wave equation, then objects held together by the forces transmitted through these waves would exhibit Lorentz invariance as a dynamic symmetry. If these objects are elementary particles, it would explain why they can be described by Lorentz-invariant field theories. The fundamental field would somehow assume the role of the aether in pre-Einstein physics, but which, unlike the aether of pre-Einstein physics, is a quantized field. Not only would such a nonrelativistic field be Galilei invariant, but it would also establish a preferred reference system at rest with the field.

The existence of such a preferred reference system is supported by the distribution of galaxies in the universe, suggesting that matter derives its existence from a cosmological field at rest with the galaxies. The existence of a preferred reference system is also suggested to explain the faster-than-light quantum correlations in a rational way.

The most simple example for a nonrelativistic Heisenberg-type nonlinear field is given by the operator field equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_0} \nabla^2 \psi - f^2 r_0^2 \psi^\dagger \psi \psi \tag{3.1}$$

where the operators ψ and $\psi^\dagger$ obey the commutation relations

$$\begin{aligned} [\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')] &= \delta(\mathbf{r}-\mathbf{r}') \\ [\psi(\mathbf{r})\psi(\mathbf{r}')] &= [\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')] = 0 \end{aligned} \tag{3.2}$$

In (3.1), m_0 is a mass and r_0 a length. The coupling constant f will be determined from the condition of Lorentz invariance in the long-wavelength limit.

With the Hartree approximation

$$\phi^* \phi^2 = \langle \psi^\dagger \psi \psi \rangle \tag{3.3}$$

where $\phi = \langle \psi \rangle$, $\phi^* = \langle \psi^\dagger \rangle$, (3.1) becomes the nonlinear Schrödinger equation:

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m_0} \nabla^2 \phi + f^2 r_0^2 \phi^* \phi^2 \tag{3.4}$$

Putting (Madelung, 1926)

$$\begin{aligned} \phi &= A e^{iS} \\ n &= A^2 \\ \mathbf{v} &= \frac{\hbar}{m_0} \operatorname{grad} S \end{aligned}$$

one obtains from (3.4) the hydrodynamic form of the nonlinear Schrödinger equation:

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{m_0} \operatorname{grad}(V+Q) \\ &\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = 0 \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} V &= f^2 r_0^2 n \\ Q &= -\frac{\hbar^2}{2m_0} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \end{aligned} \tag{3.7}$$

Going to the long-wavelength limit, one can neglect Q against V. By superimposing on n a small disturbance n' and which results in a small velocity disturbance $\mathbf{v}$, one can linearize (3.6):

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{f^2 r_0^2}{m_0} \nabla n'$$
$$\frac{\partial n'}{\partial t} = -n \operatorname{div} \mathbf{v} \tag{3.8}$$

Putting $\theta = \operatorname{div} \mathbf{v}$, one obtains from (3.8) an equation for a compression wave

$$-\frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2} + \nabla^2 \theta = 0 \tag{3.9}$$

with

$$c^2 = \frac{f^2 r_0^2 n}{m_0} \tag{3.10}$$

where c is the propagation velocity of the wave. Demanding relativistic invariance in the long-wavelength limit would then simply mean that the propagation velocity must be equal to the velocity of light. The wave equation (3.9) is Lorentz invariant under Lorentz transformations if c is set equal to the velocity of light. For dimensional reasons, we may put $n = 1/r_0^3$, by which (3.10) becomes

$$c^2 = f^2 / m_0 r_0 \tag{3.11}$$

Making, furthermore, the choice

$$m_0 r_0 c = \hbar \tag{3.12}$$

one has

$$f^2 = \hbar c \tag{3.13}$$

With this choice of the coupling constant f, Lorentz invariance is valid in the long-wavelength limit of low energies.

4. FINITISTIC FIELD EQUATION

To translate the field equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_0} \nabla^2 \psi + \hbar c r_0^2 \psi^\dagger \psi \psi \tag{4.1}$$

into a finite-difference equation, one has to use the operators Δ_0^N and Δ_1^N. Henceforth, calling $\Delta_0^1 \equiv \Delta_0$, $\Delta_1^1 \equiv \Delta_1$, $\Delta_0^3 = D_0$, $\Delta_1^3 = D_1$, one has to make the substitutions

$$\frac{\partial \psi}{\partial t} \to \Delta_1 \psi \qquad\qquad \nabla^2 \psi \to D_1^2 \psi \tag{4.2}$$

If in the first of these l is set equal to the smallest time interval t_0, one has because of (2.7)

$$\Delta_1 \psi = \frac{2}{t_0} \sinh\left(\frac{t_0}{2} \frac{\partial}{\partial t}\right) \psi \tag{4.3}$$

whereby the energy operator becomes

$$E = i\hbar \Delta_1 \tag{4.4}$$

For the replacement of the differential operator for the space part, we have to solve (2.13) for $N=3$:

$$\left[\frac{1}{\partial^2} \frac{d}{d\partial}\left(\partial^2 \frac{d}{d\partial}\right) - 3\left(\frac{r_0}{2}\right)^2\right] D_0(\partial) = 0 \tag{4.5}$$

With the condition $\lim_{r_0 \to 0} D_0 = 1$, one finds

$$D_0 = \frac{\sinh[\sqrt{3}(r_0/2)\partial]}{\sqrt{3}(r_0/2)\partial} \tag{4.6}$$

and, therefore, because of (2.14)

$$D_1 = \left(\frac{2}{r_0}\right)^2 \frac{dD_0}{d\partial_i} = \left(\frac{2}{r_0}\right)^2 \frac{dD_0}{d\partial} \frac{d\partial}{d\partial_i} = \left(\frac{2}{r_0}\right)^2 \frac{dD_0}{d\partial} \frac{\partial_i}{\partial} \tag{4.7}$$

where $\partial_i = \{\partial/\partial x, \partial/\partial y, \partial/\partial z\} = \{\partial_x, \partial_y, \partial_z\}$, and $\partial = (\partial_x^2 + \partial_y^2 + \partial_z^2)^{1/2}$.

Explicitly one finds

$$D_1 = \left(\frac{2}{r_0}\right)^2 \left\{\cosh\left[\sqrt{3}\left(\frac{r_0}{2}\right)\partial\right] - \frac{\sinh[\sqrt{3}(r_0/2)\partial]}{\sqrt{3}(r_0/2)\partial}\right\} \frac{\partial_i}{\partial^2} \tag{4.8}$$

Expanding the bracket in (4.8) up to third-order terms, using

$$\cosh x = 1 + x^2/2 + \cdots$$

$$\sinh x = x + x^3/6 + \cdots$$

one finds as required that

$$\lim_{r_0 \to 0} \mathrm{D}_1 = \partial_i \tag{4.9}$$

With the replacement of ∂_i by D_1, the quantum mechanical momentum operator $\mathbf{p} = (\hbar/i)\, \partial/\partial\mathbf{q}$ should be replaced by

$$\mathbf{p} = \frac{\hbar}{i} D_1 \tag{4.10}$$

To ensure the integrity of the classical Poisson bracket relation $\{q, p\} = 1$, a change in the momentum operator $\mathbf{p}$ must be accompanied by a change in the position operator $\mathbf{q}$. If in the quantum mechanical commutation relation

$$pq - qp = \hbar/i \tag{4.11}$$

$\mathbf{p}$ is given by (4.10), the position operator $\mathbf{q}$ must be given by[2]

$$\mathbf{q} = \left(\frac{\partial_i}{D_1}\right)\mathbf{r} \tag{4.12}$$

and for which $\lim_{r_0 \to 0} \mathbf{q} = \mathbf{r}$.

The commutation relation for the field operators, (3.2), is thereby changed into

$$[\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')] = D(|\mathbf{r} - \mathbf{r}'|) \tag{4.13}$$

where, in accordance with the finite-difference calculus, $D(|\mathbf{r} - \mathbf{r}'|)$ is a generalized three-dimensional delta function for which

$$\lim_{r_0 \to 0} D(|\mathbf{r} - \mathbf{r}'|) = \delta(|\mathbf{r} - \mathbf{r}'|) \tag{4.14}$$

In Leibniz's operator notation one may formally put

$$\int = \frac{1}{d} \tag{4.15}$$

resulting in the operator equation

$$1/(d/dx) = (1/d)\, dx = \int dx \tag{4.16}$$

[2]For the connection of the position operator with the position eigenfunction, see the Appendix.

Applied to a one-dimensional delta function, one has

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = \frac{1}{d}\,\delta(x)\, dx = \frac{1}{d/dx}\,\delta(x) = 1 \tag{4.17}$$

Since for finite-difference operations d/dx is replaced by D_1, the equation corresponding to (4.17) is

$$\frac{1}{D_1} D(x) = 1 \tag{4.18}$$

and for the three-dimensional D function occurring in (4.13)

$$\frac{1}{D_1^3} D(|\mathbf{r}-\mathbf{r}'|) = 1 \tag{4.19}$$

If one replaces in D_1 the operator symbol $\partial_i = \partial/\partial x_i$ by the equivalent operator symbol $(\int \partial x_i)^{-1}$, and thereafter expand D_1^{-3} into a power series of the operator $\int \partial x_i$, (4.19) then consists of an infinite number of integrations (very much as D_1 consists of an infinite number of differentiations).

Making the replacement (4.2), we find that the field equation (4.1) translated into a finitistic form is

$$i\hbar\Delta_1\psi = -\frac{\hbar^2}{2m_0} D_1^2\psi + \hbar c r_0^2 \psi^\dagger \psi\psi \tag{4.20}$$

With the help of (3.12), and furthermore assuming that $c = r_0/t_0$, one can bring (4.20) into a form in which it depends only on the parameters r_0 and t_0:

$$it_0\Delta_1\psi = -\tfrac{1}{2} r_0^2 D_1^2\psi + r_0^3 \psi^\dagger \psi\psi \tag{4.21}$$

To obtain the finitistic form for the quantum mechanical equation of motion of an operator F

$$\frac{dF}{dt} = \frac{i}{\hbar}(HF - FH) \equiv \frac{i}{\hbar}[H, F] \tag{4.22}$$

where H is the Hamilton operator, the rhs remains unchanged, because the Poisson bracket for any dynamical quantity can be reduced to a sum of Poisson brackets for position and momentum, whereas in the lhs the operator d/dt has to be replaced by Δ_1. The equation of motion (4.22) is therefore changed into

$$\Delta_1 F = \frac{i}{\hbar}[H, F] \tag{4.23}$$

5. LAGRANGE FORMALISM

The proposed finitistic field theory is also fully consistent with the Lagrange formalism, provided we replace everywhere the space and time differentiation operators by finite-difference operators. For an unquantized classical field, the operators ψ and $\psi^\dagger$ are replaced by the functions ϕ and ϕ^*, and one has for the Lagrange density

$$\mathscr{L} = i\hbar\phi^*\Delta_1\phi - \frac{\hbar^2}{2m_0}(D_1\phi^*)(D_1\phi) - \tfrac{1}{2}\hbar c r_0^2(\phi^*\phi)^2 \tag{5.1}$$

Variation with regard to ϕ^* according to

$$\frac{\partial\mathscr{L}}{\partial\phi^*} - D_1\frac{\partial\mathscr{L}}{\partial(D_1\phi^*)} = 0 \tag{5.2}$$

results in the finitistic classical field equation

$$i\hbar\Delta_1\phi = -\frac{\hbar^2}{2m_0}D_1^2\phi + \hbar c r_0^2\phi^*\phi^2 \tag{5.3}$$

If the variation is carried out with regard to ϕ, according to

$$\frac{\partial\mathscr{L}}{\partial\phi} - D_1\frac{\partial\mathscr{L}}{\partial(D_1\phi)} - \Delta_1\frac{\partial\mathscr{L}}{\partial(\Delta_1\phi)} = 0 \tag{5.4}$$

one obtains the conjugate complex equation of (5.3) by replacing $\phi \to \phi^*$, $\Delta_1\phi \to -\Delta_1\phi^*$.

The momentum density canonically conjugate to ϕ is

$$\pi = \frac{\partial\mathscr{L}}{\partial(\Delta_1\phi)} = i\hbar\phi^* \tag{5.5}$$

and hence the Hamilton density

$$\begin{aligned}\mathbf{H} &= \pi\Delta_1\phi - \mathscr{L}\\ &= -\frac{i\hbar}{2m_0}(D_1\pi)(D_1\phi) - \frac{i}{2}c r_0^2\phi^*\phi^2\pi\\ &= \frac{\hbar^2}{2m_0}(D_1\phi^*)(D_1\phi) + \tfrac{1}{2}\hbar c r_0^2(\phi^*\phi)^2\end{aligned} \tag{5.6}$$

We are now in a position to prove that the unquantized and quantized field equations formally agree. Because of (4.23), the operator ψ obeys the equation of motion

$$i\hbar\Delta_1\psi=[\psi, H] \tag{5.7}$$

where

$$H=D_1^{-3}\mathbf{H} \tag{5.8}$$

with D_1^{-3} the finite-difference volume integration operator satisfying (4.19).

We therefore must have

$$\begin{aligned} i\hbar\Delta_1\psi=&\left[\psi, D_1^{-3'}\frac{\hbar^2}{2m_0}(D_1'\psi^{\dagger'})(D_1'\psi')\right]\\ &+[\psi, D_1^{-3'}\tfrac{1}{2}\hbar c r_0^2\psi^{\dagger'}\psi'\psi^{\dagger'}\psi'] \end{aligned} \tag{5.9}$$

To evaluate commutators in (5.9), we have to use (4.13). In the first commutator we obtain by partial (finite-difference operator) integration

$$\begin{aligned} [\psi, D_1^{-3'}(D_1'\psi^{\dagger'})\cdot(D_1'\psi')]&=-[\psi, D_1^{-3'}\psi^{\dagger'}D_1^{2'}\psi']\\ &=-D_1^{-3'}[\psi', \psi^{\dagger'}]D_1^{2'}\psi'\\ &=-D_1^{-3'}D_1^{2'}\psi' D(|\mathbf{r}-\mathbf{r}'|)=-D_1^2\psi \end{aligned} \tag{5.10}$$

For the second commutator, the integrands have the form

$$\begin{aligned} &\psi\psi^{\dagger'}\psi'\psi^{\dagger'}\psi'-\psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi\\ &=\psi^{\dagger'}\psi\psi'\psi^{\dagger'}\psi'+D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger'}\psi'-\psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi\\ &=\psi^{\dagger'}\psi'\psi\psi^{\dagger'}\psi'-\psi^{\dagger'}\psi'\psi^{\dagger'}\psi'\psi+D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger'}\psi'\\ &=\psi^{\dagger'}\psi'[\psi\psi^{\dagger'}\psi'-\psi^{\dagger'}\psi'\psi]+D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger'}\psi'\\ &=\psi^{\dagger'}\psi'\psi' D(|\mathbf{r}-\mathbf{r}'|)+D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger'}\psi'\\ &=2\psi^{\dagger'}\psi'\psi' D(|\mathbf{r}-\mathbf{r}'|) \end{aligned} \tag{5.11}$$

Multiplying (5.11) by $D_1^{-3'}$ then leads to $2\psi^\dagger\psi\psi$. Inserting the results of (5.10) and (5.11) into (5.9), one obtains the operator field equation (4.20), which shows that the classical and quantum equations have the same form.

Finally, we may prove that our finitistic field theory conserves the number of particles. With the particle number operator

$$N_\pm=D_1^{-3}\psi^\dagger\psi \tag{5.12}$$

one has to show that

$$i\hbar\Delta_1 N = [N, H] = 0 \tag{5.13}$$

Since it is well known that for a nonrelativistic field theory without interaction, but in the presence of an external potential, the particle number is conserved, we have only to show that this is also true if the nonlinear self-interaction term is included. In the commutator it leads to integrals with integrands of the form [and which can be transformed using (4.13)]

$$\begin{aligned}
&\psi^\dagger\psi\psi^{\dagger\prime}\psi'\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi^{\dagger\prime}\psi'\psi^\dagger\psi \\
&= \psi^\dagger\psi^{\dagger\prime}\psi\psi'\psi^{\dagger\prime}\psi' + \psi^\dagger D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi^{\dagger\prime}\psi'\psi^\dagger\psi \\
&= \psi^{\dagger\prime}\psi^\dagger\psi'\psi\psi^{\dagger\prime}\psi' + \psi^\dagger D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi^{\dagger\prime}\psi'\psi^\dagger\psi \\
&= \psi^{\dagger\prime}\psi'\psi^\dagger\psi\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}D(|\mathbf{r}-\mathbf{r}'|)\psi\psi^{\dagger\prime}\psi' + \psi^\dagger D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger\prime}\psi' \\
&\quad - \psi^\dagger\psi'\psi^{\dagger\prime}\psi'\psi^\dagger\psi \\
&= \psi^{\dagger\prime}\psi'[\psi^\dagger\psi\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi^\dagger\psi] + \psi^\dagger D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger\prime}\psi' \\
&\quad - \psi^{\dagger\prime}D(|\mathbf{r}-\mathbf{r}'|)\psi\psi^{\dagger\prime}\psi'
\end{aligned} \tag{5.14}$$

In (5.14) the last two terms cancel upon integration (multiplication with D_1^{-3}). The first term is zero as well, because it can be reduced to a term which would arise in (5.13) in the presence of an externally applied potential. It would contain the term $\psi^\dagger\psi\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi^\dagger\psi$, and is of the same form as the square bracket in (5.14). It therefore follows that the number of particles is conserved.

6. MAXIMUM ENERGY AND MOMENTUM

For energies $E \ll m_0c^2$, we can approximate the finite-difference field equation (4.20) by (4.1) and the commutation relation (4.13) by (3.2). In this approximation the field equation leads to a phonon–roton spectrum below the scale m_0c^2, as in the theory of superfluidity. In the vicinity of $E \simeq m_0c^2$, and where the approximation becomes invalid, departures from this energy spectrum can be expected. It limits the dimensionality of the Hilbert space and thereby eliminates all divergences. To study this departure, we may omit the nonlinear self-coupling term. If the free field is limited in the dimensionality of the Hilbert space, the self-coupling cannot alter this fact.

The finite-difference field equation

$$i\hbar\Delta_1\psi = -\frac{\hbar^2}{2m_0}D_1^2\psi \tag{6.1}$$

has for a wavefield $\psi=\psi(x, t)$ the form

$$\frac{2i\hbar}{t_0}\sinh\left(\frac{t_0}{2}\frac{\partial}{\partial t}\right)\psi$$
$$=-\frac{8m_0c^2}{r_0^2}\left\{\cosh\left[\sqrt{3}\left(\frac{r_0}{2}\right)\frac{\partial}{\partial x}\right]-\frac{\sinh[\sqrt{3}(r_0/2)\,\partial/\partial x]}{\sqrt{3}(r_0/2)\,\partial/\partial x}\right\}^2\frac{1}{(\partial/\partial x)^2}\psi \quad (6.2)$$

For a plane wave

$$\psi=A\,e^{i(kx-\omega t)} \quad (6.3)$$

(6.2) leads to the dispersion relation

$$\sin\left(\frac{\omega t_0}{2}\right)=\frac{3}{[\sqrt{3}(r_0/2)k]^2}\left\{\frac{\sin[\sqrt{3}(r_0/2)k]}{\sqrt{3}(r_0/2)k}-\cos\left[\sqrt{3}\left(\frac{r_0}{2}\right)k\right]\right\}^2 \quad (6.4)$$

Putting

$$x=\sqrt{3}(r_0/2)k \quad (6.5)$$

one can write instead of (6.4)

$$\begin{aligned}&[3\sin(\omega t_0/2)]^{1/2}=f(x)\\ &f(x)=\frac{3}{x}\left(\frac{\sin x}{x}-\cos x\right)\end{aligned} \quad (6.6)$$

In the limit $x\to 0$, one has $f(x)\to x$, and for $x\to\infty$, $f(x)\to 0$. The function $f(x)$ has a maximum at $x\simeq 2.1$, where $f(2.1)\simeq 1.3$. We therefore find that

$$k_{\max}\simeq 2.4/r_0 \quad (6.7)$$

and

$$\sin(\omega_{\max}t_0/2)\simeq 0.57 \quad (6.8)$$

The maximum energy is then computed with the energy operator (4.4):

$$\begin{aligned}E_{\max}&=(2\hbar/t_0)\sin(\omega_{\max}t_0/2)\\ &=2m_0c^2\sin(\omega_{\max}t_0/2)\\ &\simeq 1.14m_0c^2\end{aligned} \quad (6.9)$$

From (6.8) we also have

$$\omega_{\max}t_0\simeq 1.22 \quad (6.10)$$

and hence

$$\frac{\omega_{max}}{k_{max}} = 0.56 \frac{r_0}{t_0} = 0.56c \tag{6.11}$$

Neglecting the nonlinear term in the nonfinitistic field equation (4.1), we find that the energy spectrum there is given by

$$E = \hbar^2 k^2 / 2m_0 \tag{6.12}$$

It is unbounded because even in a Galilei-invariant nonfinitistic field theory without a cutoff the zero-point energy is divergent.

7. FINITISTIC FIELD THEORY AS A MODEL FOR A UNIFIED THEORY OF ELEMENTARY PARTICLES

It has been shown that a nonlinear field theory of the form (4.1) leads to a vortex sponge of quantized vortices, and that it can explain both Maxwell's electromagnetic and Einstein's gravitational vacuum field equations as collective excitations of this vortex sponge (Kelly, 1976; Winterberg, 1990). However, it cannot explain Dirac spinors. Furthermore, with the mass m_0 set equal the Planck mass, the zero-point vacuum energy is $\sim 10^{95}$ g/cm^3. This large mass density would lead to large gravitational fields, which are obviously not observed. To compensate the large mass density of the zero-point energy, one could in principle introduce a large cosmological constant, but this is a procedure which is not very satisfactory. The problem, however, can be overcome if physical reality also includes the countable set of all negative integers, which together with the natural numbers (respectively positive integers) have the same cardinal number $\aleph_0$ as the natural numbers alone. This suggests that the fundamental field equation must have two components. It has been shown that the two-component nonlinear operator field equation

$$i\hbar \frac{\partial \psi_\pm}{\partial t} = \mp \frac{\hbar^2}{2m_0} \nabla^2 \psi_\pm + 2\hbar c r_0^2 (\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-) \psi_\pm \tag{7.1}$$

can eliminate the large zero-point mass density of the vacuum and also leads to Dirac spinors (Winterberg, 1988, 1991).

Translated into its finitistic form, (7.1) becomes

$$it_0 \Delta_1 \psi_\pm = \mp \tfrac{1}{2} r_0^2 D_1^2 \psi_\pm + 2r_0^3 [\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-] \psi_\pm \tag{7.2}$$

with (7.1) the long-wavelength approximation of (7.2).

In the approximation (7.1), the field equation can be viewed as an assembly of densely packed Planck masses of both signs, resembling a model for the vacuum suggested by Sakharov (1968), who postulated the existence of "ghost particles" to compensate the large mass density of the positive Planck masses.

8. FINITISTIC FIELD THEORY AND CANTOR'S CONTINUUM HYPOTHESIS

At Cantor's time it was generally believed that besides matter, there must be an aether, qualitatively different from matter, and filling all of space. Assuming that physical space has the same structure as mathematical space, Cantor came up with the strange hypothesis that all material objects are made up from $\aleph_0$ mass monads, and the aether atoms are made up from the infinitely larger number of $\aleph_1$ aether monads (Cantor, 1932). Cantor's hypothesis appears to have some relation to the proposed finitistic field theory, as can be seen as follows: The finitistic field theory is motivated by the assumption that physics should be described by countable numbers only, having the cardinal number $\aleph_0$. But in formulating such a finitistic field theory, the field equations can only be expressed as differential equations of infinite order. Since a differential equation of any order, and certainly of infinite order, covers both the countable and the noncountable numbers, which have the cardinal number $\aleph_0 + \aleph_1 = \aleph_1$, the infinite-order differential equation is therefore reminiscent of Cantor's idea that the aether atoms are made up of $\aleph_1$ aether monads. And by going to the long-wavelength limit where the field equation can be approximated by a Schrödinger equation for all Planck masses, the Planck masses are reminiscent of Cantor's mass monads.

In its two-valuedness, the finitistic field equation (7.2) is representative for the $SU2$ group, which is also the rotation group in a three-dimensional space of constant curvature. It has been conjectured by von Weizsäcker (1971) that the three dimensions of space suggest a two-valuedness of the fundamental field. Assuming that physical space has constant curvature and is finite, the number of discrete steps of length r_0 which can be taken in space in making a round trip is given by

$$N \sim R/r_0 \tag{8.1}$$

where R is the curvature radius of this space. The number N, of course, is much smaller than $\aleph_0$. One might therefore entertain the conjecture that within a Planck radius step, space can be subdivided into N discrete points, which only in the limit $R \to \infty$ becomes equal to $\aleph_0$. The smallest measurable distance therefore would be

$$r_{\min} \sim r_0/N \sim r_0^2/R \tag{8.2}$$

With $r_0^2 = G\hbar/c^3$ (G is Newton's constant) and $R \sim GM/c^2$ (M is the mass of the universe), one finds that

$$r_{\min} \sim \hbar/Mc \tag{8.3}$$

which is the Compton wavelength of the universe. The smallest distance in a finite universe, which was guessed by a number-theoretic conjecture, therefore turns out to be equal to the smallest distance which can be measured by quantum mechanics, using the energy of the entire universe. With $r_0 \sim 10^{-33}$ cm and $R \sim 10^{28}$ cm, one obtains $r_{\min} \sim 10^{-94}$ cm.

APPENDIX

With $\psi_n(q)$ the (one-dimensional) position eigenfunction and q the position operator, the eigenvalues q_n and eigenfunctions are determined by the equation

$$q\psi_n(q) = q_n\psi_n(q) \tag{A.1}$$

If the position can be precisely measured, one would have

$$\psi_n(q) = \delta(q - q_n) \tag{A.2}$$

Inserting (A.2) into the lhs of (A.1) and integrating from $q = -\infty$ to $q = +\infty$, one has

$$\int_{-\infty}^{+\infty} q\delta(q - q_n)\, dq = q_n \tag{A.3}$$

Using Leibniz's operator notation $\int = 1/d$, one can also write for (A.3)

$$\frac{dq}{d} \cdot q \cdot \delta(q - q_n) = q_n \tag{A.4}$$

For $q = q_n$ one has

$$\frac{dq}{d}\,\delta(q - q_n) = 1 \tag{A.5}$$

Now, if the position eigenfunction is instead given by the generalized delta function $D(q-q_n)$, which was defined by (4.18) such that

$$\frac{1}{D_1} D(q-q_n)=1 \tag{A.6}$$

it follows by comparison with (A.4) and (A.5) that with the position eigenfunction given by $D(q-q_n)$, the position operator must be replaced by putting

$$q \rightarrow \frac{1}{D_1}\left(\frac{d}{dq}\right) \cdot q \tag{A.7}$$

which is equivalent to (4.12).

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